

# Boundary crossing probabilities for diffusions with piecewise linear drifts

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## Abstract

In this work we study boundary crossing probabilities of general one-dimensional diffusion processes with respect to constant boundaries. We first give an explicit formula of the probabilities for diffusions with piecewise linear drift and constant diffusion coefficient in terms of its Laplace transform. Then we provide an approximation scheme to obtain the boundary crossing probabilities for more general diffusions. Finally, an estimate of convergence rate of the approximation is also given.

**Keywords:** Boundary crossing probability; First passage density; First passage time; Diffusions with piecewise linear drifts

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# 1 Introduction

Boundary crossing probabilities for stochastic processes play an important role in many research fields. For example, in business and industry a company's financial status can be represented by its total asset and debt at any given time, which can be regarded as two stochastic processes. Then the company goes bankrupt when its asset process falls below its debt process for the first time. Therefore the company's default risk can be described by the probability that the difference of the asset and debt processes reaches a certain threshold. In finance, the arbitrage-free price of a barrier option is given by the expectation of the payoff function of strike price and the first hitting time of the underlying asset price to the barrier. In neuroscience, a popular integrate-and-fire model assumes that a neuron fires a spike when the membrane potential reaches a threshold. In engineering, the system reliability can be represented by the probability that a damage process crossing a threshold. In epidemiology, if the spread of an infectious disease is modeled by a stochastic process, then it is of interest and importance to know the probability that this process reaches a certain threshold within a certain time limit. The first passage time also arises in many other disciplines such as biology, chemistry, ecology, economics, environmental science, genetics, physics as well as statistics. Some references for applications can be found in Wang and Pötzelberger (2007) [12]. More recent literature includes Touboul and Faugeras (2007) [10], Taillefumier and Magnasco (2010) [9], Molini et al (2011) [6].

Given its important role in many applications, however, the computation of boundary crossing probabilities is a challenging task. Explicit formulas exist for only a few special instances. For example, the reflection principle of one dimensional Brownian motion yields an explicit formula for it to cross a constant boundary. Further applying the reflection principle in depth leads to explicit formulas for linear and piecewise linear boundary crossing probabilities ([3, 1, 11]). In general it is difficult to obtain explicit formulas for a stochastic process crossing even the simplest constant boundary. However, in real applications, the underlying processes are usually described by more general diffusion processes than Brownian motion, while the later is used to represent random perturbation in a system. Therefore, it is important to study the

boundary crossing probability for general diffusion processes. An attempt is made by Wang and Pötzelberger (2007) [12], who proposed a transformation approach to compute the boundary crossing probability for a class of diffusions which can be expressed as piecewise monotone functional of standard Brownian motion. For this kind of diffusions, explicit formulae for the boundary crossing probabilities for certain boundaries can be obtained. However, from both theoretical and practical points of view this class of processes is very limited. More specifically, [12] consider the following diffusion process

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x,$$

where  $(W_t)_{t \geq 0}$  is the standard Brownian motion on  $\mathbf{R}$ , and show that under some regular conditions on  $\mu$  and  $\sigma$   $X_t$  can be expressed as a monotone functional of standard Brownian motion if and only if the drift coefficient  $\mu$  and diffusion coefficient  $\sigma$  satisfy the following second-order differential equation:

$$\frac{\partial}{\partial x} \left[ \frac{1}{\sigma} \frac{\partial \sigma}{\partial t} + \sigma \frac{\partial}{\partial x} \left( \frac{1}{2} \frac{\partial \sigma}{\partial t} - \frac{\mu}{\sigma} \right) \right] = 0.$$

For instance, when  $\sigma$  is a constant,  $\mu(t, x)$  must be a linear function of  $x$  in order for the above differential equation to hold. Hence, the class of processes satisfying the above equation is very limited. Moreover, it seems difficult, if at all possible, to generalize this transformation method to obtain the boundary crossing probabilities for other diffusion processes that are not a functional of Brownian motion. This motivates us to find a novel approach to this problem.

To simplify notation in this paper we will focus on the diffusion processes satisfying the following stochastic differential equation (SDE) with additive noise:

$$dX_t = \mu(X_t)dt + dW_t, \quad X_0 = x. \tag{1.1}$$

There is no loss of generality in the sense that any diffusion process satisfying

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{1.2}$$

with differentiable and non-zero  $\sigma(x)$  can be transformed into one with unit diffusion coefficient through the transformaiton

$$F(y) = \int_{y_0}^y \frac{1}{\sigma(u)} du$$

for some  $y_0$ . Indeed, it is easy to verify by Ito's formula that the process  $Y_t := F(X_t)$  satisfies SDE (1.1) with drift coefficient  $\mu(x) = (\mu/\sigma - \sigma'/2) \circ F^{-1}(x)$ .

The idea is to first obtain the explicit formula of Laplace transform of boundary crossing probability for a diffusion process with piecewise linear drift, and then use this formula to approximate the corresponding probability for more general processes. In our approach, an estimate of the convergence rate of the approximation can also be derived. More precisely, consider the diffusion  $X_t$  with continuous and piecewise linear drift  $\mu(x)$  in the form:

$$\mu(x) = \sum_{i=-\infty}^{\infty} \mathbf{1}_{[x_{i-1}, x_i]}(x)(a_i x + b_i), \quad (1.3)$$

where  $\sup\{|a_i| + |b_i|; i = 0, \pm 1, \pm 2, \dots\} < \infty$  and  $\{\dots < x_{-m} < \dots < x_{-1} < x_0 < x_1 < \dots < x_m < \dots\}$  is a partition of  $\mathbf{R}$ . We show that the first passage time

$$\tau_c = \inf\{t > 0; X_t \geq c\} \quad (1.4)$$

admits a density  $f_c(t)$  on  $[0, \infty)$  and an explicit formula for its Laplace transform can be obtained using the method of [2]. Then for a diffusion process  $X_t$  satisfying (1.1) with  $\mu(x)$  admitting bounded derivative, we construct a sequence of processes  $(X_t^{(n)})_{t \geq 0}$  defined by

$$dX_t^{(n)} = \mu_n(X_t^{(n)})dt + dW_t, \quad X_0^{(n)} = x \quad (1.5)$$

with continuous piecewise linear drifts  $\mu_n$  for  $n \in \mathbf{N}$  such that  $\sup_{x \in \mathbf{R}} |\mu(x) - \mu_n(x)| < C/n$  for some constant  $C > 0$ . Finally, we show that  $X_t^{(n)}$  converges to  $X_t$  almost surely for each  $t > 0$ , and further the boundary crossing probability of  $X_t^{(n)}$  converges to that of  $X_t$  in the sense of

$$\mathbf{P}\left(\sup_{s \leq t} (X_s^{(n)} - c) < 0\right) \longrightarrow \mathbf{P}\left(\sup_{s \leq t} (X_s - c) < 0\right), \text{ as } \varepsilon \rightarrow 0.$$

In addition, we show that the rate of this convergence is of order  $1/n$ .

## 2 The process with piecewise linear drift

In this section, we first prove that the boundary crossing density exists for a diffusion with piecewise linear drift (1.3). Then we give the explicit formula of its Laplace transform using the method of Darling and Siegert [2].

Given a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , let  $x \in \mathbf{R}$  and  $(W_t^x)_{t \geq 0}$  be a Brownian motion starting at  $W_0^x = x$ . For the simplicity of notation, subsequently we will write  $W_t^x = W_t$ . Let  $C_x([0, \infty))$  be the continuous path space starting at  $w_0 = x$ , i.e.

$$C_x([0, \infty)) = \{w : [0, \infty) \rightarrow \mathbf{R} \text{ continuous; } w_0 = x\}.$$

Let  $\mathbf{Q}_x$  be the law of Brownian motion  $(W_t)_{t \geq 0}$  starting at  $W_0 = x$  in  $C_x([0, \infty))$ . Further, define  $\mathcal{F}_t = \sigma(w_s; 0 \leq s \leq t)$ . By standard theory of stochastic differential equations (eg, [5]), there exists a unique non-explosive solution  $(X_t)_{t \geq 0}$  to the SDE (1.1) for  $\mu(x)$  being in the form (1.3). In order to prove that  $\tau_c$  has a density w.r.t. the Lebesgue measure, we shall generalize the result of Downes and Borovkov [4, Theorem 3.1] to the case where  $\mu(x)$  is not necessarily everywhere differentiable.

**Lemma 2.1** *Let  $p(s, x, y)$  denote the transition density of the process  $(X_s)_{s \geq 0}$  and  $q(s, x, y)$  the transition density of the Brownian motion. Let  $\mathbf{P}_x$  denote the law of  $(X_s)_{s \geq 0}$  starting at  $X_0 = x$ , and  $\mathbf{P}_x^z$  the law of  $(X_s)_{0 \leq s \leq t}$  starting at  $X_0 = x$  and conditioning on  $X_t = z$ . Similarly, let  $\mathbf{Q}_x$  and  $\mathbf{Q}_x^z$  denote the corresponding probability measure with respect to the Brownian motion  $(W_s)_{s \geq 0}$ . Then for  $t > 0$  and every  $A \in \mathcal{F}_t$ ,*

$$\mathbf{P}_x^z(A) \leq \frac{q(t, x, z)}{p(t, x, z)} e^{G(z) - G(x) - 3Mt/2} \mathbf{Q}_x^z(A), \quad (2.1)$$

where  $G(y) = \int_{y_0}^y \mu(z) dz$  for some fixed  $y_0$ , and  $M = \inf_{y \in \mathbf{R}} \{\mu(y)^2 + \frac{1}{3}\mu'_-(y)\}$  with  $\mu'_-(y) = \liminf_{z \rightarrow y} \frac{\mu(z) - \mu(y)}{z - y}$ .

**Proof.** Since  $X_t = W_t + \int_0^t \mu(X_s) ds$  and  $\mu$  is bounded, by Girsanov's theorem,

$$\mathbf{P}_x(A) = \mathbf{Q}_x(\zeta_t \mathbf{1}_A), \quad A \in \mathcal{F}_t, \quad (2.2)$$

where  $Q_x(\zeta_t \mathbf{1}_A) = \int \zeta_t \mathbf{1}_A dQ_x$  and

$$\zeta_t = \exp \left[ \int_0^t \mu(X_s) dW_s - \frac{1}{2} \int_0^t \mu(X_s)^2 ds \right].$$

Conditioning on the value of corresponding processes at time  $t$ , equation (2.2) can be written as

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbf{P}_x^z(A) p(t, x, z) dz &= \int_{-\infty}^{\infty} \mathbf{P}_x^z(A) \mathbf{P}_x(X_t \in dz) \\ &= \int_{-\infty}^{\infty} \mathbf{Q}_x^z(\zeta_t \mathbf{1}_A) \mathbf{Q}_x(W_t \in dz) = \int_{-\infty}^{\infty} \mathbf{Q}_x^z(\zeta_t \mathbf{1}_A) q(t, x, z) dz. \end{aligned}$$

By the arbitrariness of  $A \in \mathcal{F}_t$ , we have

$$\mathbf{P}_x^z(A) = \frac{q(t, x, z)}{p(t, x, z)} \mathbf{Q}_x^z(\zeta_t \mathbf{1}_A). \quad (2.3)$$

Further, let  $\rho(y) = 1/\sqrt{2\pi}e^{-y^2/2}$  and  $\mu_n(y) = \int_{-\infty}^{\infty} \frac{n\mu(z)}{\sqrt{2\pi}} e^{-\frac{n^2(z-y)^2}{2}} dz$  for  $n \in \mathbf{N}$ . Then  $\mu_n(y)$  converges to  $\mu(y)$  for every  $y \in \mathbf{R}$  as  $n \rightarrow \infty$ . Moreover, by the dominated convergence theorem,

$$\mathbf{E} \left( \int_0^t (\mu_n(X_s) - \mu(X_s)) dW_s \right)^2 = \mathbf{E} \int_0^t (\mu_n(X_s) - \mu(X_s))^2 ds \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

It follows that, up to an extraction of subsequence,  $\int_0^t \mu_n(X_s) dW_s \rightarrow \int_0^t \mu(X_s) dW_s$  almost surely. Similarly by the dominated convergence theorem,  $G_n(X_t)$  converges almost surely to  $G(X_t)$  as  $n \rightarrow \infty$ , where  $G_n(y) = \int_y^{\infty} \mu_n(z) dz$ . Further, by Fatou's lemma, we have

$$\begin{aligned} \mu'_n(y) &= \lim_{h \rightarrow 0} \frac{\mu_n(y+h) - \mu_n(y)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_{-\infty}^{\infty} \left( \mu(y+h-\frac{z}{n}) - \mu(y-\frac{z}{n}) \right) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ &\geq \int_{-\infty}^{\infty} \liminf_{h \rightarrow 0} \frac{\mu(y-\frac{z}{n}+h) - \mu(y-\frac{z}{n})}{h} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ &= \int_{-\infty}^{\infty} \mu'_-(y-\frac{z}{n}) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz. \end{aligned}$$

Since  $\mu$  is a piecewise linear function, and  $\mu'_-(y) = \liminf_{z \rightarrow y} \frac{\mu(z) - \mu(y)}{z-y}$ , it's easy to get

$$\liminf_{n \rightarrow \infty} \mu'_n(y) \geq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} \mu'_-(y-\frac{z}{n}) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \geq \mu'_-(y), \quad \forall y \in \mathbf{R}.$$

Hence,  $\lim_{n \rightarrow \infty} \int_0^t \mu'_n(X_s) ds \geq \int_0^t \mu'_-(X_s) ds$  almost surely. Applying Itô's formula to  $G_n(X_t)$ , we have

$$\int_0^t \mu_n(X_s) dW_s = G_n(X_t) - G_n(X_0) - \int_0^t \mu_n(X_s)^2 ds - \frac{1}{2} \int_0^t \mu'_n(X_s) ds.$$

Passing to the limit as  $n \rightarrow \infty$  yields

$$\int_0^t \mu(X_s) dW_s \leq G(X_t) - G(X_0) - \int_0^t \mu(X_s)^2 ds - \frac{1}{2} \int_0^t \mu'_-(X_s) ds, \quad a.s. \quad (2.4)$$

Finally, combining (2.4) with (2.3) we have

$$\mathbf{P}_x^z(A) \leq \frac{q(t, x, z)}{p(t, x, z)} e^{G(z) - G(x)} \mathbf{Q}_x^z \left( e^{-\frac{3}{2} \int_0^t \mu(X_s)^2 ds - \frac{1}{2} \int_0^t \mu'_-(X_s) ds} \mathbf{1}_A \right),$$

which yields the desired result (2.1). ■

**Lemma 2.2** For  $x, z < c$ ,  $h > 0$ , it holds

$$\mathbf{P}_x \left( \sup_{0 \leq s \leq t} (X_s - c) < 0 | X_t = z \right) \leq \frac{q(t, x, z)}{p(t, x, z)} e^{G(z) - G(x) - 3Mt/2} \left( 1 - e^{-\frac{2(c-x)(c-z)}{t}} \right), \quad (2.5)$$

and

$$\mathbf{P}_x \left( \sup_{t \leq s \leq t+h} (X_s - c) \geq 0 | X_t = z \right) \leq 1 - \Phi \left( \frac{-\mu_+ h + c - z}{\sqrt{h}} \right) + e^{2\mu_+(c-z)} \Phi \left( \frac{-\mu_+ h - c + z}{\sqrt{h}} \right), \quad (2.6)$$

where  $\mu_+ = \sup_{y \in \mathbf{R}} \mu(y)$ ,  $\Phi(y) = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$ , and  $M, G(\cdot)$  are defined in Lemma 2.1.

**Proof.** (i) By Anderson [1] or Siegmund [8] (P. 375), we have

$$\mathbf{Q}_x^z \left( \sup_{0 \leq s \leq t} (W_s - c) < 0 \right) = 1 - e^{-\frac{2(c-x)(c-z)}{t}}.$$

Substituting the above into (2.1) results in (2.5).

(ii) Define a new process

$$d\hat{X}_s = \mu_+ ds + dW_s, \quad \hat{X}_t = z.$$

Then according to the comparison theorem ( Ikeda and Watanabe [5] ), conditioning on  $X_t = z$ ,  $X_s \leq \hat{X}_s$  for every  $s \geq t$  almost surely. Therefore

$$\begin{aligned}
& \mathbf{P}_x \left( \sup_{t \leq s \leq t+h} (X_s - c) \geq 0 | X_t = z \right) \\
& \leq \mathbf{P} \left( \sup_{t \leq s \leq t+h} (\hat{X}_s - c) \geq 0 \right) \\
& = \mathbf{P} \left( \sup_{t \leq s \leq s+h} (\mu_+(s-t) + W_s - W_t + z - c) \geq 0 \right) \\
& = \mathbf{Q}_0 \left( \sup_{0 \leq s \leq h} (W_s + \mu_+ s + z - c) \geq 0 \right) \\
& = 1 - \Phi \left( \frac{-\mu_+ h + c - z}{\sqrt{h}} \right) + e^{2\mu_+(c-z)} \Phi \left( \frac{-\mu_+ h - c + z}{\sqrt{h}} \right),
\end{aligned}$$

where the last equality follows from Anderson [1]. This yields (2.6).  $\blacksquare$

**Theorem 2.3 (Existence of density)** *Let  $(X_t)_{t \geq 0}$  be defined by (1.1) with  $\mu(\cdot)$  satisfying (1.3) and  $\tau_c$  defined by (1.4). Then  $\tau_c$  has a density  $f_c(t, x)$  satisfying*

$$f_c(t, x) \leq \frac{c-x}{\sqrt{2\pi t^{3/2}}} e^{G(c)-G(x)-\frac{3Mt}{2}} e^{-\frac{(c-x)^2}{2t}}, \quad (2.7)$$

where  $x < c$  is the initial state of the process  $(X_t)_{t \geq 0}$ .

**Proof.** In order to show the existence of the density of  $\tau_c$ , by Radon-Nikodym theorem we only need to prove that, for every  $h > 0$ ,

$$\lim_{h \downarrow 0} \mathbf{P}_x(\tau_c \in (t, t+h))/h < \infty,$$

where

$$\begin{aligned}
\mathbf{P}_x(\tau_c \in (t, t+h)) &= \int_{-\infty}^c \mathbf{P}_x(\tau_c \in (t, t+h) | X_t = z) \mathbf{P}_x(X_t \in dz) \\
&= \int_{-\infty}^{c-h^{1/4}} + \int_{c-h^{1/4}}^c \mathbf{P}_x(\tau_c \in (t, t+h) | X_t = z) \mathbf{P}_x(X_t \in dz) \\
&=: \mathbf{I}_1 + \mathbf{I}_2
\end{aligned}$$



To this end, we estimate  $I_1$  and  $I_2$  subsequently. First, due to (2.6) we have

$$\begin{aligned} I_1 &= \int_{-\infty}^{c-h^{1/4}} \mathbf{P}_x \left( \sup_{0 \leq s \leq t} (X_s - c) < 0 | X_t = z \right) \mathbf{P}_x \left( \sup_{t \leq s \leq t+h} (X_s - c) \geq 0 | X_t = z \right) \mathbf{P}_x(X_t \in dz) \\ &\leq \int_{-\infty}^{c-h^{1/4}} \mathbf{P}_x \left( \sup_{t \leq s \leq t+h} (X_s - c) \geq 0 | X_t = z \right) \mathbf{P}_x(X_t \in dz) \\ &\leq \int_{-\infty}^c 1 - \Phi \left( \frac{-\mu_+ h + h^{1/4}}{\sqrt{h}} \right) + e^{2\mu_+(c-z)} \Phi \left( \frac{-\mu_+ h - h^{1/4}}{\sqrt{h}} \right) \mathbf{P}_x(X_t \in dz) \end{aligned}$$

Now we show that  $\int_{-\infty}^c e^{2\mu_+(c-z)} \mathbf{P}_x(X_t \in dz) < \infty$ . Let  $\tilde{X}_s$  be the solution of the SDE

$$d\tilde{X}_s = \mu_- ds + dW_s, \quad \tilde{X}_0 = x,$$

where  $\mu_- = \inf_{y \in \mathbf{R}} \mu(y)$  which is finite by assumption. Then  $\tilde{X}_t \leq X_t$  almost surely for every  $t > 0$ . Hence, for each  $\lambda < 0$ ,

$$\mathbf{E}(e^{\lambda X_t}) \leq \mathbf{E}(e^{\lambda \tilde{X}_t}) = \mathbf{E}(e^{\lambda \mu_- t + \lambda W_t}) < \infty.$$

By a similar argument, we have for each  $\lambda > 0$ ,

$$\mathbf{E}(e^{\lambda X_t}) \leq \mathbf{E}(e^{\lambda \mu_+ t + \lambda W_t}) < \infty.$$

Thus, we have

$$\int_{-\infty}^c e^{2\mu_+(c-z)} \mathbf{P}_x(X_t \in dz) \leq e^{2\mu_+ c} \mathbf{E}[e^{2\mu_+ X_t}] < \infty.$$

Moreover, since

$$1 - \Phi \left( \frac{-\mu_+ h + h^{1/4}}{\sqrt{h}} \right) = o(h), \quad \Phi \left( \frac{-\mu_+ h - h^{1/4}}{\sqrt{h}} \right) = o(h), \quad \text{as } h \downarrow 0,$$

it follows that  $I_1 = o(h)$  as  $h \downarrow 0$ , where  $o(h)$  is the little-o notation which means  $o(h)/h \rightarrow 0$  as  $h \rightarrow 0$ .

Next we estimate  $I_2$ . Using Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
I_2 &= \int_{c-h^{1/4}}^c \mathbf{P}_x \left( \sup_{0 \leq s \leq t} (X_s - c) < 0 | X_t = z \right) \mathbf{P}_x \left( \sup_{t \leq s \leq t+h} (X_s - c) \geq 0 | X_t = z \right) \mathbf{P}_x(X_t \in dz) \\
&\leq \int_{c-h^{1/4}}^c q(t, x, z) e^{G(z) - G(x) - \frac{3Mt}{2}} (1 - e^{-2(c-x)(c-z)/t}) \\
&\quad \cdot \left[ 1 - \Phi \left( \frac{-\mu_+ h + c - z}{\sqrt{h}} \right) + e^{2\mu_+(c-z)} \Phi \left( \frac{-\mu_+ h - c + z}{\sqrt{h}} \right) \right] dz \\
&= \frac{2(c-x)}{t} \int_0^{h^{1/4}} e^{G(c-y) - G(x) - \frac{3Mt}{2}} (y + O(y^2)) \left[ 1 - \Phi(-\mu_+ h^{1/2} + y h^{-1/2}) \right. \\
&\quad \left. + e^{2\mu_+ y} \Phi(-\mu_+ h^{1/2} - y h^{-1/2}) \right] q(t, x, c-y) dy,
\end{aligned}$$

where  $O(y^2)$  is the big-O notation. It follows that

$$\begin{aligned}
\lim_{h \downarrow 0} I_2/h &\leq K \lim_{h \downarrow 0} \frac{1}{h} \int_0^{h^{1/4}} (y + O(y^2)) \left[ 1 - \Phi(-\mu_+ h^{1/2} + y h^{-1/2}) \right. \\
&\quad \left. + e^{2\mu_+ y} \Phi(-\mu_+ h^{1/2} - y h^{-1/2}) \right] q(t, x, c-y) dy,
\end{aligned} \tag{2.8}$$

where  $K = \frac{2(c-x)}{t} e^{G(c) - G(x) - \frac{3Mt}{2}}$ . To obtain the limit of the right hand side of (2.8), we only need to calculate

$$\begin{aligned}
&\lim_{h \downarrow 0} \frac{1}{h} \int_0^{h^{1/4}} y \left[ 1 - \Phi(-\mu_+ h^{1/2} + y h^{-1/2}) \right. \\
&\quad \left. + e^{2\mu_+ y} \Phi(-\mu_+ h^{1/2} - y h^{-1/2}) \right] q(t, x, c-y) dy \\
&= \lim_{h \downarrow 0} \int_0^{h^{1/4}} y \frac{d}{dh} \left[ 1 - \Phi(-\mu_+ h^{1/2} + y h^{-1/2}) \right. \\
&\quad \left. + e^{2\mu_+ y} \Phi(-\mu_+ h^{1/2} - y h^{-1/2}) \right] q(t, x, c-y) dy \\
&= \lim_{h \downarrow 0} \int_0^{h^{1/4}} \frac{y}{2\sqrt{2\pi}} \left[ e^{-\frac{(\mu_+ h^{1/2} + y h^{-1/2})^2}{2}} (\mu_+ h^{-1/2} + y h^{-3/2}) \right. \\
&\quad \left. + e^{-\frac{(-\mu_+ h^{1/2} + y h^{-1/2})^2}{2}} (-\mu_+ h^{-1/2} + y h^{-3/2}) \right] q(t, x, c-y) dy \\
&= \lim_{h \downarrow 0} \int_0^{h^{-1/4}} \frac{y}{4\pi\sqrt{t}} \left[ e^{-\frac{(\mu_+ h^{1/2} + y)^2}{2}} (\mu_+ h^{1/2} + y) \right. \\
&\quad \left. + e^{-\frac{(\mu_+ h^{1/2} - y)^2}{2}} (-\mu_+ h^{1/2} + y) \right] e^{-\frac{(c-x-yh^{1/2})^2}{2t}} dy \\
&= \frac{1}{2\sqrt{2\pi t}} e^{-\frac{(c-x)^2}{2t}}.
\end{aligned}$$

Substituting this into (2.8), we have

$$\lim_{h \downarrow 0} I_2/h \leq \frac{c-x}{\sqrt{2\pi t^{3/2}}} e^{G(c)-G(x)-\frac{3Mt}{2}} e^{-\frac{(c-x)^2}{2t}}.$$

Analog to the arguments on  $I_1$ , we finally obtain

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbf{P}_x(\tau_c \in (t, t+h)) \leq \frac{c-x}{\sqrt{2\pi t^{3/2}}} e^{G(c)-G(x)-\frac{3Mt}{2}} e^{-\frac{(c-x)^2}{2t}},$$

which yields the existence of the density  $f_c(t, x)$  and its upper bound (2.7).  $\blacksquare$

After proving the existence of the density for the first passage time  $\tau_c$  of diffusions with piecewise linear drift, now we use the method of Darling and Siegert [2] to obtain the distribution of  $\tau_c$  in terms of its Laplace transform. Recall that  $p(t, x, y)$  denotes the transition density of the process  $(X_t)_{t \geq 0}$  and  $f_c(t, x)$  is the density of  $\tau_c$  for the process  $(X_t)_{t \geq 0}$  starting at  $X_0 = x$ . Their Laplace transforms are defined as

$$\begin{aligned} \hat{p}(x, y|\lambda) &= \int_0^\infty e^{-\lambda t} p(t, x, y) dt, \\ \hat{f}_c(x|\lambda) &= \int_0^\infty e^{-\lambda t} f_c(t, x) dt. \end{aligned}$$

In [2], the authors study the properties of  $\hat{p}$  and  $\hat{f}_c$  under the assumption of the existence of  $f_c$  and give the following results.

**Proposition 2.4 ([2])** (i)  $\hat{p}(x, y|\lambda)$  is in a form of product

$$\hat{p}(x, y|\lambda) = \begin{cases} u(x)u_1(y) & y > x \\ v(x)v_1(y) & y < x, \end{cases} \quad (2.9)$$

and

$$\hat{f}_c(x|\lambda) = \frac{u(x)}{u(c)}, \quad x < c. \quad (2.10)$$

(ii) the functions  $u(x)$  and  $v(x)$  can be chosen as any two linearly independent solutions of the differential equation

$$\frac{1}{2} \frac{d^2 \eta}{dx^2} + \mu(x) \frac{d\eta}{dx} - \lambda \eta = 0, \quad (2.11)$$

with initial condition  $u(-\infty) = v(\infty) = 0$ .

**Proof.** The key points of the proof are based on the following basic observations:

$$p(t, x, y) = \int_0^t f_c(s, x) p(t-s, c, y) ds, \text{ for } x < c < y;$$

$$\text{and } p(t, x, y) \text{ satisfies: } \frac{\partial p}{\partial t} = \mu(x) \frac{\partial p}{\partial x} + \frac{1}{2} \sigma(x) \frac{\partial^2 p}{\partial x^2}.$$

We refer to [2, Theorem 3.1 and 4.1] for the complete argument. ■

According to the theory of ordinary differential equation, the explicit solution of equation (2.11) can be given for some special cases. Therefore one feasible method to find the first passage time distribution of a general diffusion process is to approximate this process by a kind of diffusion processes for which the corresponding equation (2.11) admits explicit solutions. Based on this idea, we establish the following main theorem which provides an explicit solution of (2.11) for a diffusion process  $(X_t)_{t \geq 0}$  defined by (1.1) and has drift (1.3).

**Theorem 2.5** *Let  $(X_t)_{t \geq 0}$  be the solution of SDE (1.1) with  $\mu$  satisfying (1.3). Then for  $x < c$ ,*

$$\hat{f}_c(x|\lambda) = \frac{u(x)}{u(c)}, \quad (2.12)$$

with

$$u(x) = \sum_{i=-\infty}^{\infty} \left[ J_i\left(-\frac{\lambda}{2a_i}, \frac{1}{2}; -a_i\left(x + \frac{b_i}{a_i}\right)^2\right) \mathbf{1}_{a_i \neq 0} + \tilde{J}_i(b_i; x) \mathbf{1}_{a_i=0} \right] \mathbf{1}_{[x_{i-1}, x_i]}(x), \quad (2.13)$$

and satisfying  $u(-\infty) = 0$ , where

$$J_i(a, \frac{1}{2}; x) = C_{1,i} \Psi(a, \frac{1}{2}; x) + C_{2,i} x^{1/2} \Psi(a + \frac{1}{2}, \frac{3}{2}; x),$$

$$\Psi(a, \frac{1}{2}; x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(\frac{1}{2})_k} \frac{x^k}{k!}, \quad (a)_k := a(a+1) \dots (a+k-1),$$

$$\tilde{J}(b_i; x) = \begin{cases} e^{-b_i x} [C_{1,i} e^{\Delta_i x/2} + C_{2,i} e^{-\Delta_i x/2}], & \text{if } \Delta_i^2 = 4b_i^2 + 8\lambda > 0 \\ e^{-b_i x} [C_{1,i} \sin(\Delta_i x/2) + C_{2,i} \cos(\Delta_i x/2)], & \text{if } \Delta_i^2 = -4b_i - 8\lambda > 0 \\ e^{-b_i x} (C_{1,i} x + C_{2,i}), & \text{if } \Delta_i^2 = 4b_i^2 + 8\lambda = 0. \end{cases}$$

Here the constants  $C_{1,i}$ ,  $C_{2,i}$  are chosen such that  $u(x)$  is differentiable. They can be determined recursively: if  $C_{1,i}$ ,  $C_{2,i}$  for all  $i \leq k$  are known, then  $C_{1,k+1}$ ,  $C_{2,k+1}$  are chosen such that

$\lim_{x \uparrow x_k} u(x) = \lim_{x \downarrow x_k} u(x)$  and  $\lim_{x \uparrow x_k} \frac{u(x) - u(x_k)}{x - x_k} = \lim_{x \downarrow x_k} \frac{u(x) - u(x_k)}{x - x_k}$ . Furthermore, they are uniquely determined up to a multiplicative constant, which will not change the value of  $\hat{f}(x|\lambda)$ .

**Proof.** The explicit solutions of the following second-order differential equations are known.

By [7, 2.1.2.11], the general solution of ODE

$$\frac{d^2 y}{dx^2} + a \frac{dy}{dx} + by = 0$$

is

$$y = \begin{cases} \exp(-\frac{1}{2}ax) [C_1 \exp(\frac{1}{2}\Delta x) + C_2 \exp(-\frac{1}{2}\Delta x)] & \text{if } \Delta^2 = a^2 - 4b > 0, \\ \exp(-\frac{1}{2}ax) [C_1 \sin(\frac{1}{2}\Delta x) + C_2 \cos(\frac{1}{2}\Delta x)] & \text{if } \Delta^2 = 4b - a^2 > 0, \\ \exp(-\frac{1}{2}ax) (C_1 x + C_2) & \text{if } a^2 = 4b, \end{cases}$$

where  $C_1, C_2$  are constants. similarly, by [7, 2.1.2.108], the general solution of ODE

$$\frac{1}{2} \frac{d^2 y}{dx^2} + (a_1 x + b_1) \frac{dy}{dx} + b_0 y = 0, \quad a_1 \neq 0$$

is  $J(b_0, \frac{1}{2}; -a_1(x + \frac{b_1}{a_1})^2)$ , where

$$J(a, b; x) = C_1 \Psi(a, b; x) + C_2 x^{1-b} \Psi(a - b + 1, 2 - b; x),$$

$$\Psi(a, b; x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}.$$

Combining them together, we get the explicit solution (2.13) to the differential equation

$$\frac{1}{2} \frac{d^2 y}{dx^2} + \mu(x) \frac{dy}{dx} - \lambda y = 0.$$

The result follows by applying Proposition 2.4 ■

**Remark 2.6** It is worthwhile to include some asymptotic relations of  $\Psi(a, b; x)$  here.

$$\Psi(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} \left[ 1 + O\left(\frac{1}{|x|}\right) \right] \quad \text{if } x \rightarrow +\infty,$$

$$\Psi(a, b; x) = \frac{\Gamma(b)}{\Gamma(b-a)} (-x)^{-a} \left[ 1 + O\left(\frac{1}{|x|}\right) \right] \quad \text{if } x \rightarrow -\infty,$$

where  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$  is the gamma function.

**Remark 2.7** Our approach to prove the above theorem can be used to deal with the first passage time of two-sided constant boundaries. Similar to our treatment of Theorem 2.3, we can prove that the first passage density for two-sided constant boundaries also exists. Then using [2, Theorem 3.2], we can get the explicit expression of the Laplace transform of the corresponding first passage time density. Inevitably, the expression in this case will be more complicated.

### 3 Approximation of diffusions with general drifts

In this section we consider the boundary crossing probability for general diffusions. The basic idea is to use diffusions with piecewise linear drifts to approximate the diffusions with general drifts. The key issue is to estimate the accuracy of this approximation.

To this end we first establish a rule to control the accuracy of the approximation. Our following result, Proposition 3.1 below, may be viewed as an extension of Downes and Borovkov's result [4, Theorem 4.1] to the situation of diffusion processes with non-differentiable drifts. Again, let  $(X_t)_{t \geq 0}$  be a diffusion process satisfying

$$dX_t = \mu(X_t)dt + dW_t, \quad X_0 = x. \quad (3.1)$$

For each  $\varepsilon > 0$ , let  $(X_t^\varepsilon)_{t \geq 0}$  be a diffusion process satisfying the SDE:

$$dX_t^\varepsilon = \mu_\varepsilon(X_t^\varepsilon)dt + dW_t, \quad X_0^\varepsilon = x. \quad (3.2)$$

Assume that

(H1)  $\mu_\varepsilon : \mathbf{R} \rightarrow \mathbf{R}$  is bounded and there exists a positive constant  $K$  such that

$$|\mu_\varepsilon(y) - \mu_\varepsilon(z)| \leq K|y - z|, \quad \forall y, z \in \mathbf{R}, \varepsilon > 0;$$

(H2)  $\sup_{y \in \mathbf{R}} |\mu(y) - \mu_\varepsilon(y)| \leq \varepsilon$ .

For  $c > x$ , define the first passage time for the process  $(X_t^\varepsilon)_{t \geq 0}$  as  $\tau_c^\varepsilon = \inf\{t > 0; X_t^\varepsilon \geq c\}$ . Further, let  $\mu_- = \inf_{y \in \mathbf{R}} \mu(y)$  and  $\mu_+ = \sup_{y \in \mathbf{R}} \mu(y)$  denote the lower and upper bound of the

function  $\mu(y)$  respectively. Obviously assumptions (H1) and (H2) imply that  $\mu_-$  and  $\mu_+$  are finite.

**Proposition 3.1** *Let  $(X_t)_{t \geq 0}$  and  $(X_t^\varepsilon)_{t \geq 0}$  be defined as in (3.1) and (3.2). Then under (H1) and (H2), for every  $T > 0$ , it holds*

$$\begin{aligned} & |\mathbf{P}(\tau_c > T) - \mathbf{P}(\tau_c^\varepsilon > T)| \\ & \leq 2Te^{3KT/2} e^{G(c)-G(x)} \left[ \int_0^T \frac{c-x}{\sqrt{2\pi}s^{3/2}} e^{-\frac{(c-x)^2}{2s}} \left( |\mu_-| + \frac{1}{\sqrt{2\pi(T-s)}} \right) ds \right] \varepsilon + o(\varepsilon), \end{aligned} \quad (3.3)$$

where  $G(y) = \int_{y_0}^y \mu(z) dz$  for some fixed  $y_0$ .

**Proof.** First we have

$$\begin{aligned} d(X_t - X_t^\varepsilon) &= (\mu(X_t) - \mu_\varepsilon(X_t^\varepsilon))dt \\ &= (\mu(X_t) - \mu_\varepsilon(X_t) + \mu_\varepsilon(X_t) - \mu_\varepsilon(X_t^\varepsilon))dt. \end{aligned}$$

By (H1) and (H2),

$$|X_t - X_t^\varepsilon| \leq \varepsilon t + \int_0^t K|X_s - X_s^\varepsilon| ds$$

and it follows from Gronwall's lemma that

$$|X_t - X_t^\varepsilon| \leq \varepsilon t e^{Kt}, \quad a.s. \quad \forall t > 0.$$

Using this inequality, we have

$$\begin{aligned} & \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t - c) < 0\right) - \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t^\varepsilon - c) < 0\right) \\ & \leq \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t^\varepsilon - \varepsilon t e^{Kt} - c) < 0\right) - \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t^\varepsilon - c) < 0\right) \\ & \leq \mathbf{P}\left(0 \leq \sup_{0 \leq t \leq T} (X_t^\varepsilon - c) < \varepsilon T e^{KT}\right) \\ & = \int_0^T \mathbf{P}(\tau_c^\varepsilon \in ds) \mathbf{P}\left(\sup_{s \leq t \leq T} (X_t^\varepsilon - c) < \varepsilon T e^{KT} \mid X_s^\varepsilon = c\right). \end{aligned} \quad (3.4)$$

Since  $\mu_-$  is finite by (H1) and (H2),

$$X_t^\varepsilon - X_s^\varepsilon \geq \mu_-(t-s) + W_t - W_s, \quad t > s,$$

which yields

$$\begin{aligned}
& \mathbf{P}\left(\sup_{s \leq t \leq T} (X_t^\varepsilon - c) < \varepsilon T e^{KT} \mid X_s^\varepsilon = c\right) \\
& \leq \mathbf{P}\left(\sup_{0 \leq t \leq T-s} (\mu_- t + W_t) < \varepsilon T e^{KT}\right) \\
& = \Phi\left(\frac{-\mu_-(T-s) + \varepsilon T e^{KT}}{\sqrt{T-s}}\right) - e^{2\mu_- \varepsilon T e^{KT}} \Phi\left(\frac{-\mu_-(T-s) - \varepsilon T e^{KT}}{\sqrt{T-s}}\right) \\
& =: I(T, s, K, \mu_-; \varepsilon).
\end{aligned} \tag{3.5}$$

Substituting the above inequality into (3.4) and using Theorem 2.3, we obtain

$$\begin{aligned}
& \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t - c) < 0\right) - \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t^\varepsilon - c) < 0\right) \\
& \leq \int_0^T \frac{c-x}{\sqrt{2\pi s^{3/2}}} e^{G_\varepsilon(c) - G_\varepsilon(x) - \frac{3M_\varepsilon s}{2}} e^{-\frac{(c-x)^2}{2s}} I(T, s, K, \mu_-; \varepsilon) ds,
\end{aligned} \tag{3.6}$$

where  $G_\varepsilon(y) = \int_{y_0}^y \mu_\varepsilon(z) dz$ ,  $M_\varepsilon = \inf\{\mu_\varepsilon^2 + \frac{1}{3}\mu'_{\varepsilon,-}(y)\}$ , and

$$\mu'_{\varepsilon,-}(y) := \liminf_{z \rightarrow y} \frac{\mu_\varepsilon(z) - \mu_\varepsilon(y)}{z - y} \leq K.$$

Further, by the inequality  $1 - e^{-x} \leq x$  for all  $x \in \mathbf{R}$ , we have the following upper bound for the term  $I(T, s, K, \mu_-; \varepsilon)$ :

$$I(T, s, K, \mu_-; \varepsilon) \leq \left(\frac{2T e^{KT}}{\sqrt{2\pi(T-s)}} + 2|\mu_-|T e^{KT}\right) \varepsilon. \tag{3.7}$$

In order to prove the inverse direction of the inequality, we write

$$\begin{aligned}
& \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t^\varepsilon - c) < 0\right) - \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t - c) < 0\right) \\
& \leq \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t^\varepsilon - c) < 0\right) - \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t^\varepsilon + \varepsilon T e^{KT} - c) < 0\right) \\
& \leq \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t^\varepsilon + \varepsilon T e^{KT} - c) < \varepsilon T e^{KT}\right) - \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t^\varepsilon + T e^{KT} - c) < 0\right) \\
& = \int_0^T \mathbf{P}(\tau_{\tilde{c}}^\varepsilon \in ds) \mathbf{P}\left(\sup_{s \leq t \leq T} (X_t^\varepsilon - \tilde{c}) < \varepsilon T e^{KT} \mid X_s^\varepsilon = \tilde{c}\right),
\end{aligned}$$

where  $\varepsilon$  is small enough so that  $c - \varepsilon T e^{KT} > x$  as  $c > x$ , and  $\tilde{c} = c - \varepsilon T e^{KT}$ . Using the same arguments for (3.5) and (3.6), we can obtain the lower bound. Combining this with (3.7) and



noting that  $M_\varepsilon \geq -K/3$ , we have

$$\begin{aligned} & \left| \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t - c) < 0\right) - \mathbf{P}\left(\sup_{0 \leq t \leq T} (X_t - c) < 0\right) \right| \\ & \leq 2Te^{3KT/2} e^{G(c)-G(x)} \left[ \int_0^T \frac{c-x}{\sqrt{2\pi}s^{3/2}} e^{-\frac{(c-x)^2}{2s}} \left( |\mu_-| + 1/(2\pi(T-s))^{\frac{1}{2}} \right) ds \right] \varepsilon + o(\varepsilon), \end{aligned}$$

which concludes the proof.  $\blacksquare$

Let  $X_t$  be the solution of SDE (3.1). Assume that there exist positive constants  $M_1, M_2$  such that

$$\sup_{y \in \mathbf{R}} |\mu(y)| \leq M_1, \quad \sup_{x, y \in \mathbf{R}} |\mu(x) - \mu(y)| \leq M_2|x - y|. \quad (3.8)$$

For each  $n \in \mathbf{N}$ , we partition the real line  $(-\infty, \infty)$  with sub-intervals with endpoints  $\{0, \pm\frac{1}{n}, \pm\frac{2}{n}, \dots\}$ .

Let  $\mu_n(y)$  be the piecewise linear function taking values at  $x_i = \frac{i}{n}$  ( $i \in \mathbf{Z}$ ) to be  $\mu(x_i)$ . Thus, for each  $i \in \mathbf{Z}$ ,

$$\begin{aligned} \max_{y \in [x_i, x_{i+1}]} |\mu(y) - \mu_n(y)| & \leq \max_{y \in [x_i, x_{i+1}]} \{ \max\{|\mu(y) - \mu(x_i)|, |\mu(y) - \mu(x_{i+1})|\} \} \\ & \leq M_2/n. \end{aligned} \quad (3.9)$$

Further, let  $X_t^{(n)}$  be the solution of the SDE

$$dX_t^{(n)} = \mu_n(X_t^{(n)})dt + dW_t, \quad X_0^{(n)} = x. \quad (3.10)$$

Then for each  $n \in \mathbf{N}$ ,  $(X_t^{(n)})_{t \geq 0}$  is a diffusion process with piecewise linear drift. For  $c > x$ , Theorem 2.5 gives the distribution of its first passage time  $\tau_c^{(n)} := \inf\{t > 0; X_t^{(n)} \geq c\}$  in terms of its Laplace transform. As we have stated at the beginning of the paper, the boundary crossing probability of the general diffusion process  $(X_t)_{t \geq 0}$  can be approximated by the boundary crossing probability of  $(X_t^{(n)})_{t \geq 0}$ . The accuracy of this approximation is given in the following theorem.

**Theorem 3.2** *Let  $(X_t)_{t \geq 0}$  and  $(X_t^{(n)})_{t \geq 0}$  be the processes defined above. Then under condition (3.8), for  $c > x$ ,  $T > 0$ , it holds*

$$\begin{aligned} & \left| \mathbf{P}(\tau_c > T) - \mathbf{P}(\tau_c^{(n)} > T) \right| \\ & \leq 2TM_2 e^{3M_2T/2} e^{M_1(c-x)} \left[ \int_0^T \frac{c-x}{\sqrt{2\pi}s^{3/2}} e^{-\frac{(c-x)^2}{2s}} \left( M_1 + \frac{1}{\sqrt{2\pi(T-s)}} \right) ds \right] \frac{1}{n} + o\left(\frac{1}{n}\right). \end{aligned} \quad (3.11)$$

**Proof.** By the definition of  $\mu_n(y)$ , it is easy to see that

$$\sup_{y \in \mathbf{R}} |\mu_n(y)| \leq M_1, \quad |\mu_n(y) - \mu_n(z)| \leq M_2|y - z|, \quad y, z \in \mathbf{R}. \quad (3.12)$$

It follows that  $G(c) - G(x) \leq M_1(c - x)$ . Therefore (3.11) follows from Proposition 3.1 and (3.9). ■

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